FIVE-PARAMETER FAMILY OF PARTIAL DIFFERENTIAL SYSTEMS IN TWO VARIABLES

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Abstract. We find a five-parameter family of partial differential systems in two variables with two polynomial Hamiltonians. We give its symmetry and holomorphy conditions. These symmetries, holomorphy conditions and invariant divisors are new.

1. Introduction

In this paper, we present a 5-parameter family of partial differential systems in two variables explicitly given by

$$\begin{aligned} dq_1&=\frac{\partial H_1}{\partial p_1}dt+\frac{\partial H_2}{\partial p_1}ds, \quad dp_1=-\frac{\partial H_1}{\partial q_1}dt-\frac{\partial H_2}{\partial q_1}ds,\\ dq_2&=\frac{\partial H_1}{\partial p_2}dt+\frac{\partial H_2}{\partial p_2}ds, \quad dp_2=-\frac{\partial H_1}{\partial q_2}dt-\frac{\partial H_2}{\partial q_2}ds \end{aligned} \tag{1}$$

with the polynomial Hamiltonians:

(2)
$$\begin{aligned} H_1=&H_{VI}(q_1,p_1,t;\alpha_1,\alpha_2,\alpha_3,\alpha_4)\\ &+\alpha_2p_2\left\{\frac{-(t-1)sq_1+t(s-1)q_2+(t-s)q_1q_2}{t(t-1)(t-s)}+\frac{(q_1-t)q_2(q_2-1)}{t(t-1)(t-\eta)}\right\}\\ &+\alpha_5p_1\left\{\frac{(t-s)q_1(q_1-1)+t(t-1)(q_1-q_2)}{t(t-1)(t-s)}+\frac{(q_1-t)((t-1)q_1+(q_1-t)q_2)}{t(t-1)(t-\eta)}\right\}\\ &-p_1p_2\left\{\frac{(t-1)(sq_1^2+tq_2^2)-(t-s)q_2(q_1^2+t)-2t(s-1)q_1q_2}{t(t-1)(t-s)}-\frac{(q_1-t)^2q_2(q_2-1)}{t(t-1)(t-\eta)}\right\}\\ &+\frac{\alpha_2\alpha_5(2tq_1-q_1-tq_2+q_1q_2-\eta q_1)}{t(t-1)(t-\eta)},\\ H_2=&\pi(H_1), \end{aligned}$$

where the transformation π is explicitly given by

$$\begin{aligned} \pi :&(q_1,p_1,q_2,p_2,t,s;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\\ &\rightarrow(q_2,p_2,q_1,p_1,s,t;\alpha_0,\alpha_1,\alpha_5,\alpha_3,\alpha_4,\alpha_2). \end{aligned} \tag{3}$$

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Here q_1, p_1, q_2 and p_2 denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_5$ are complex parameters satisfying the relation:

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 = 1. \tag{4}$$

This parameter's relation can be obtained by holomorphy conditions in Theorem 2.1.

The symbol $H_{VI}(q, p, t; \beta_1, \beta_2, \beta_3, \beta_4)$ denotes the Hamiltonian of the second-order Painlevé VI equations (see [7]) given by

$$\begin{aligned} &t(t-1)(t-\eta)H_{VI}(q, p, t; \beta_1, \beta_2, \beta_3, \beta_4) \\ &= q(q-1)(q-\eta)(q-t)p^2 \\ &+ \{\beta_1(t-\eta)q(q-1) + 2\beta_2 q(q-1)(q-\eta) \\ &+ \beta_3(t-1)q(q-\eta) + \beta_4 t(q-1)(q-\eta)\}p \\ &+ \beta_2\{(\beta_1+\beta_2)(t-\eta) + \beta_2(q-1) \\ &+ \beta_3(t-1) + t\beta_4\}q \quad (\beta_0 + \beta_1 + 2\beta_2 + \beta_3 + \beta_4 = 1, \quad \eta \in \mathbb{C} - \{0, 1\}). \end{aligned} \tag{5}$$

We give its symmetry and holomorphy conditions. These symmetries, holomorphy conditions and invariant divisors are new.

After we review the notion of accessible singularity and local index, we make its holomorphy conditions by resolving the accessible singularities.

2. Symmetry and holomorphy conditions

In this section, we give its symmetry and holomorphy conditions. These symmetries, holomorphy conditions and invariant divisors are new.

Theorem 2.1. *Let us consider a polynomial Hamiltonian system with Hamiltonians $H_i \in \mathbb{C}(t, s)[q_1, p_1, q_2, p_2]$ $(i = 1, 2)$. We assume that*

(A1) $deg(H_i) = 6$ with respect to q_1, p_1, q_2, p_2.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i, $i = 0, 1, \dots, 5$:

$$\begin{aligned} &r_0 : x_0 = -p_1((q_1 - t)p_1 + (q_2 - s)p_2 - \alpha_0),\ y_0 = \frac{1}{p_1},\ z_0 = (q_2 - s)p_1,\ w_0 = \frac{p_2}{p_1}, \\ &r_1 : x_1 = -p_1((q_1 - \eta)p_1 + (q_2 - \eta)p_2 - \alpha_1),\ y_1 = \frac{1}{p_1},\ z_1 = (q_2 - \eta)p_1,\ w_1 = \frac{p_2}{p_1}, \\ &r_2 : x_2 = \frac{1}{q_1},\ y_2 = -q_1(q_1 p_1 + \alpha_2),\ z_2 = q_2,\ w_2 = p_2, \\ &r_3 : x_3 = -p_1((q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_3),\ y_3 = \frac{1}{p_1},\ z_3 = (q_2 - 1)p_1,\ w_3 = \frac{p_2}{p_1}, \\ &r_4 : x_4 = -p_1(q_1 p_1 + q_2 p_2 - \alpha_4),\ y_4 = \frac{1}{p_1},\ z_4 = q_2 p_1,\ w_4 = \frac{p_2}{p_1}, \\ &r_5 : x_5 = q_1,\ y_5 = p_1,\ z_5 = \frac{1}{q_2},\ w_5 = -(q_2 p_2 + \alpha_5)q_2. \end{aligned} \tag{6}$$

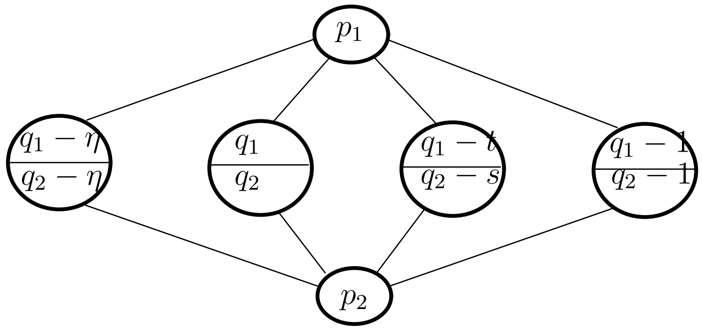

FIGURE 1. The symbol in each circle denotes the invariant cycle for the system.

Then such a system coincides with the system (1) *with two polynomial Hamiltonians* (2).

PROPOSITION 2.2. *In each coordinate $r_i,\ i=0,1,\dots,5$, the Hamiltonians H_{j1} and H_{j2} on $U_j \times B$ are expressed as a polynomial in x_j,y_j,z_j,w_j and a rational function in t and s, and satisfy the following conditions*:

$$
\begin{aligned}
&dq_1 \wedge dp_1 + dz \wedge dp_2 - dH_1 \wedge dt - dH_2 \wedge ds\\
&= dx_j \wedge dy_j + dz_j \wedge dw_j - dH_{j1} \wedge dt - dH_{j2} \wedge ds \quad (j=1,2,\dots,5),\\
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - d(H_1-p_1) \wedge dt - d(H_2-p_2) \wedge ds\\
&= dx_0 \wedge dy_0 + dz_0 \wedge dw_0 - dH_{01} \wedge dt - dH_{02} \wedge ds.
\end{aligned} \tag{7}
$$

codimension	invariant cycles	parameter's relation
1	$f_2 := p_1$	$\alpha_2 = 0$
1	$f_5 := p_2$	$\alpha_5 = 0$
2	$f_0^{(1)} := q_1 - t,\ f_0^{(2)} := q_2 - s$	$\alpha_0 = 0$
2	$f_1^{(1)} := q_1 - \eta,\ f_1^{(2)} := q_2 - \eta$	$\alpha_1 = 0$
2	$f_3^{(1)} := q_1 - 1,\ f_3^{(2)} := q_2 - 1$	$\alpha_3 = 0$
2	$f_4^{(1)} := q_1,\ f_4^{(2)} := q_2$	$\alpha_4 = 0$

We note that when $\alpha_2 = 0$, we see that the system (1) admits a particular solution $f_2 = 0$, and when $\alpha_0 = 0$, we see that the system (1) admits a particular solution $f_0^{(1)} = f_0^{(2)} = 0$.

3. Bäcklund transformations

Theorem 3.1. *The system* (1) *admits the following transformations as its Bäcklund transformations*: with the notation $(*)=(q_1,p_1,q_2,p_2,\eta,t,s;\alpha_0,\alpha_1,\dots,\alpha_5)$,

(8)

$$s_1:(*)\rightarrow\left(q_1+\frac{\alpha_2}{p_1},p_1,q_2,p_2,\eta,t,s;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4+\alpha_2,\alpha_5\right),$$

$$s_2:(*)\rightarrow\left(q_1,p_1,q_2+\frac{\alpha_5}{p_2},p_2,\eta,t,s;\alpha_0+\alpha_5,\alpha_1+\alpha_5,\alpha_2,\alpha_3+\alpha_5,\alpha_4+\alpha_5,-\alpha_5\right),$$

$$\begin{aligned}
\pi_1:(*)\rightarrow&\Big(\frac{\eta-q_1}{\eta-1},-(\eta-1)p_1,\frac{\eta-q_2}{\eta-1},-(\eta-1)p_2,\frac{\eta}{\eta-1},\frac{\eta-t}{\eta-1},\frac{\eta-s}{\eta-1};\\
&\alpha_0,\alpha_4,\alpha_2,\alpha_3,\alpha_1,\alpha_5\Big),\\
\pi_2:(*)\rightarrow&\Big(\frac{q_1(t-\eta)}{t-q_1+tq_1-\eta t},-\frac{(t-q_1+tq_1-\eta t)\{(t-q_1+tq_1-\eta t)p_1+\alpha_2(t-1)\}}{t(t-\eta)(\eta-1)},\\
&\frac{q_2(s-\eta)}{s-q_2+sq_2-\eta s},-\frac{(s-q_2+sq_2-\eta s)\{(s-q_2+sq_2-\eta s)p_2+\alpha_5(s-1)\}}{s(s-\eta)(\eta-1)},\\
&\eta,\frac{\eta-t}{1-2t+\eta t},\frac{\eta-s}{1-2s+\eta s};\alpha_3,\alpha_1,\alpha_2,\alpha_0,\alpha_4,\alpha_5\Big),\\
\pi_3:(*)\rightarrow&\Big(\frac{(t-1)q_1}{t-q_1-\eta t+\eta tq_1},\frac{(t-q_1+\eta t(q_1-1))\{(q_1-t)p_1+\alpha_2-\eta t((q_1-1)p_1+\alpha_2)\}}{t(t-1)(\eta-1)},\\
&\frac{(s-1)q_2}{s-q_2-\eta s+\eta sq_2},\frac{(s-q_2+\eta s(q_2-1))\{(q_2-s)p_2+\alpha_5-\eta s((q_2-1)p_2+\alpha_5)\}}{s(s-1)(\eta-1)},\\
&\frac{1}{\eta},\frac{\eta(t-1)}{t-\eta-\eta t+\eta^2t},\frac{\eta(s-1)}{s-\eta-\eta s+\eta^2 s};\alpha_1,\alpha_0,\alpha_2,\alpha_3,\alpha_4,\alpha_5\Big),\\
\pi_4:(*)\rightarrow&(1-q_1,-p_1,1-q_2,-p_2,1-\eta,1-t,1-s;\alpha_0,\alpha_1,\alpha_2,\alpha_4,\alpha_3,\alpha_5),\\
\pi_5:(*)\rightarrow&(q_2,p_2,q_1,p_1,\eta,s,t;\alpha_0,\alpha_1,\alpha_5,\alpha_3,\alpha_4,\alpha_2).
\end{aligned}$$

The Bäcklund transformations s_1,s_2 are determined by the invariant divisors (2).

4. Accessible singularity and local index

Let us review the notion of *accessible singularity*. Let B be a connected open domain in $\mathbb{C}$ and $\pi:\mathcal{W}\longrightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H}\subset\mathcal{W}$ is a normal crossing divisor which is flat over B. Let us consider a rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\tilde{v}\in H^0(\mathcal{W},\Theta_{\mathcal{W}}(-\log\mathcal{H})(\mathcal{H})).$$

Fixing $t_0\in B$ and $P\in\mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1,\dots,x_n)$ of $\mathcal{W}_{t_0}$ centered at P such that $\mathcal{H}_{\text{smooth}}$ can be defined by the local equation $x_1=0$. Since $\tilde{v}\in H^0(\mathcal{W},\Theta_{\mathcal{W}}(-\log\mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P=(0,\dots,0,t_0)$

as follows:

$$\tilde{v}=\frac{\partial}{\partial t}+g_1\frac{\partial}{\partial x_1}+\frac{g_2}{x_1}\frac{\partial}{\partial x_2}+\cdots+\frac{g_n}{x_1}\frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$\frac{dx_1}{dt}=g_1(x_1,\dots,x_n,t),\ \frac{dx_2}{dt}=\frac{g_2(x_1,\dots,x_n,t)}{x_1},\cdots,\frac{dx_n}{dt}=\frac{g_n(x_1,\dots,x_n,t)}{x_1}. \tag{9}$$

Here $g_i(x_1,\dots,x_n,t),\ i=1,2,\dots,n,$ are holomorphic functions defined near $P=(0,\dots,0,t_0)$.

Definition 4.1. With the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfies the condition

$$(A)\quad \tilde{v}\in H^0(\mathcal{W},\Theta_{\mathcal{W}}(-\log\mathcal{H})(\mathcal{H})).$$

We say that $\tilde{v}$ has an *accessible singularity* at $P=(0,\dots,0,t_0)$ if

$$x_1=0 \text{ and } g_i(0,\dots,0,t_0)=0 \text{ for every } i,\ 2\leq i\leq n.$$

If $P\in\mathcal{H}_{\mathrm{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_{t_0}$ over $t=t_0$. If $P\in\mathcal{H}_{\mathrm{smooth}}$ is an accessible singularity, there may be a solution of (9) which passes through P and goes into the interior $\mathcal{W}-\mathcal{H}$ of $\mathcal{W}$.

Here we review the notion of *local index*. Let v be an algebraic vector field with an accessible singular point $\overrightarrow{p}=(0,\dots,0)$ and $(x_1,\dots,x_n)$ be a coordinate system in a neighborhood centered at $\overrightarrow{p}$. Assume that the system associated with v near $\overrightarrow{p}$ can be written as

(10)

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\\\vdots\\x_{n-1}\\x_n\end{pmatrix}=\frac{1}{x_1}\left\{\begin{bmatrix}a_{11}&0&0&\dots&0\\a_{21}&a_{22}&0&\dots&0\\\vdots&\vdots&\ddots&0&0\\a_{(n-1)1}&a_{(n-1)2}&\dots&a_{(n-1)(n-1)}&0\\a_{n1}&a_{n2}&\dots&a_{n(n-1)}&a_{nn}\end{bmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_{n-1}\\x_n\end{pmatrix}+\begin{pmatrix}x_1h_1(x_1,\dots,x_n,t)\\h_2(x_1,\dots,x_n,t)\\\vdots\\h_{n-1}(x_1,\dots,x_n,t)\\h_n(x_1,\dots,x_n,t)\end{pmatrix}\right\},$$

$$(h_i\in\mathbb{C}(t)[x_1,\dots,x_n],\ a_{ij}\in\mathbb{C}(t))$$

where h_1 is a polynomial which vanishes at $\overrightarrow{p}$ and $h_i,\ i=2,3,\dots,n$ are polynomials of order at least 2 in $x_1,x_2,\dots,x_n$, We call ordered set of the eigenvalues $(a_{11},a_{22},\cdots,a_{nn})$ *local index* at $\overrightarrow{p}$.

We are interested in the case with local index

$$(1,a_{22}/a_{11},\dots,a_{nn}/a_{11})\in\mathbb{Z}^n. \tag{11}$$

These properties suggest the possibilities that a_1 is the residue of the formal Laurent series:

$$y_1(t)=\frac{a_{11}}{(t-t_0)}+b_1+b_2(t-t_0)+\cdots+b_n(t-t_0)^{n-1}+\cdots\quad(b_i\in\mathbb{C}), \tag{12}$$

and the ratio $(1, a_{22}/a_{11}, \dots, a_{nn}/a_{11})$ is resonance data of the formal Laurent series of each $y_i(t)$ $(i = 2, \dots, n)$, where $(y_1, \dots, y_n)$ is original coordinate system satisfying $(x_1, \dots, x_n) = (f_1(y_1, \dots, y_n), \dots, f_n(y_1, \dots, y_n))$, $f_i(y_1, \dots, y_n) \in \mathbb{C}(t)(y_1, \dots, y_n)$.

If each component of $(1, a_{22}/a_{11}, \dots, a_{nn}/a_{11})$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

The α-test,

$$t = t_0 + \alpha T, \quad x_i = \alpha X_i, \quad \alpha \rightarrow 0, \tag{13}$$

yields the following reduced system:

$$\frac{d}{dT}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \frac{1}{X_1}\begin{bmatrix} a_{11}(t_0) & 0 & 0 & \dots & 0 \\ a_{21}(t_0) & a_{22}(t_0) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_{(n-1)1}(t_0) & a_{(n-1)2}(t_0) & \dots & a_{(n-1)(n-1)}(t_0) & 0 \\ a_{n1}(t_0) & a_{n2}(t_0) & \dots & a_{n(n-1)}(t_0) & a_{nn}(t_0) \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix}, \tag{14}$$

where $a_{ij}(t_0) \in \mathbb{C}$. Fixing $t = t_0$, this system is the system of the first order ordinary differential equation with constant coefficient. Let us solve this system. At first, we solve the first equation:

$$X_1(T) = a_{11}(t_0)T + C_1 \quad (C_1 \in \mathbb{C}). \tag{15}$$

Substituting this into the second equation in (14), we can obtain the first order linear ordinary differential equation:

$$\frac{dX_2}{dT} = \frac{a_{22}(t_0)X_2}{a_{11}(t_0)T + C_1} + a_{21}(t_0). \tag{16}$$

By variation of constant, in the case of $a_{11}(t_0) \neq a_{22}(t_0)$ we can solve explicitly:

$$X_2(T) = C_2(a_{11}(t_0)T + C_1)^{\frac{a_{22}(t_0)}{a_{11}(t_0)}} + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)}{a_{11}(t_0) - a_{22}(t_0)} \quad (C_2 \in \mathbb{C}). \tag{17}$$

This solution is a single-valued solution if and only if

$$\frac{a_{22}(t_0)}{a_{11}(t_0)} \in \mathbb{Z}.$$

In the case of $a_{11}(t_0) = a_{22}(t_0)$ we can solve explicitly:

$$X_2(T) = C_2(a_{11}(t_0)T + C_1) + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)\mathrm{Log}(a_{11}(t_0)T + C_1)}{a_{11}(t_0)} \quad (C_2 \in \mathbb{C}). \tag{18}$$

This solution is a single-valued solution if and only if

$$a_{21}(t_0) = 0.$$

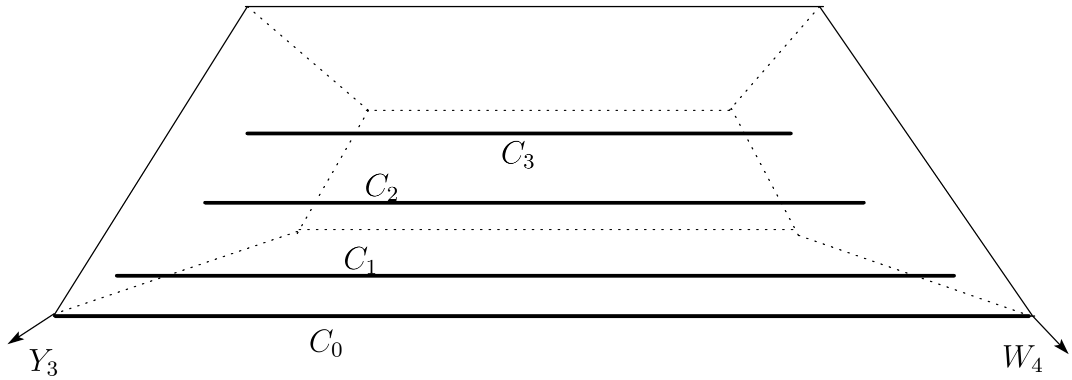

FIGURE 2. This figure denotes the boundary divisor $\mathcal{H}$ of $\mathcal{S}$. The bold lines C_i $i=0,1,2,3$ in $\mathcal{H}$ denote the accessible singular loci of the system (1).

Of course, $\frac{a_{22}(t_0)}{a_{11}(t_0)}=1\in\mathbb{Z}$. In the same way, we can obtain the solutions for each variables $(X_3,\dots,X_n)$. The conditions $\frac{a_{jj}(t)}{a_{11}(t)}\in\mathbb{Z},\ (j=2,3,\dots,n)$ are necessary condition in order to have the Painlevé property.

5. Construction of the holomorphy conditions

In this section, we will give the holomorphy conditions r_i $(i=0,1,\dots,5)$ by resolving some accessible singular loci of the system (1).

In order to consider the singularity analysis for the system (1), as a compactification of $\mathbb{C}^4$ which is the phase space of the system (1), we take 4-dimensional complex manifold $\mathcal{S}$ given in the paper [7]. This manifold can be considered as a generalization of the Hirzebruch surface.

We easily see that the rational vector field $\tilde{v}$ associated with the system (1) satisfies the condition:

$$\tilde{v}\in H^0(\mathcal{S},\Theta_{\mathcal{S}}(-\log\mathcal{H})(\mathcal{H})).$$

LEMMA 5.1. *The rational vector field $\tilde{v}$ has the following accessible singular loci* (see figure 2):

$$\left\{\begin{aligned}C_0=&\{(X_3,Y_3,Z_3,W_3)|X_3=t,Z_3=s,Y_3=0\}\\&\cup\{(X_4,Y_4,Z_4,W_4)|X_4=t,Z_4=s,W_4=0\}\cong\mathbb{P}^1,\\C_1=&\{(X_3,Y_3,Z_3,W_3)|X_3=\eta,Z_3=\eta,Y_3=0\}\\&\cup\{(X_4,Y_4,Z_4,W_4)|X_4=\eta,Z_4=\eta,W_4=0\}\cong\mathbb{P}^1,\\C_2=&\{(X_3,Y_3,Z_3,W_3)|X_3=1,Z_3=1,Y_3=0\}\\&\cup\{(X_4,Y_4,Z_4,W_4)|X_4=1,Z_4=1,W_4=0\}\cong\mathbb{P}^1,\\C_3=&\{(X_3,Y_3,Z_3,W_3)|X_3=Z_3=Y_3=0\}\\&\cup\{(X_4,Y_4,Z_4,W_4)|X_4=Z_4=W_4=0\}\cong\mathbb{P}^1.\end{aligned}\right.\tag{19}$$

Here, the coordinate systems (X_i,Y_i,Z_i,W_i) $(i=3,4)$ (see [7]) are explicitly given by

$$
\begin{aligned}
(X_3,Y_3,Z_3,W_3)&=\left(q_1,\frac{1}{p_1},q_2,\frac{p_2}{p_1}\right),\\
(X_4,Y_4,Z_4,W_4)&=\left(q_1,\frac{p_1}{p_2},q_2,\frac{1}{p_2}\right).
\end{aligned} \tag{20}
$$

This lemma can be proven by a direct calculation.

Next, we calculate its local index at the point $P:=\{(X_3,Y_3,Z_3,W_3)|X_3=t,Z_3=s,Y_3=W_3=0\}$.

Step 0: We make a change of variables.

$$X_3^{(1)}=X_3-t,\quad Y_3^{(1)}=Y_3,\quad Z_3^{(1)}=Z_3-s,\quad W_3^{(1)}=W_3.$$

Around the point P, we rewrite the system (1) as follows:

$$\frac{d}{dt}\begin{pmatrix}X_3^{(1)}\\Y_3^{(1)}\\Z_3^{(1)}\\W_3^{(1)}\end{pmatrix}=\frac{1}{Y_3^{(1)}}\left\{\begin{pmatrix}2&-\alpha_0&0&0\\0&1&0&0\\0&0&1&0\\0&\frac{\alpha_5}{t-s}&0&0\end{pmatrix}\begin{pmatrix}X_3^{(1)}\\Y_3^{(1)}\\Z_3^{(1)}\\W_3^{(1)}\end{pmatrix}+\cdots\right\}.$$

We see that this system has its local index $(2,1,1,0)$ at the point P.

For the remaining accessible singular loci, the local index is same.

Proposition 5.2. *If we resolve the accessible singular loci given in Lemma 5.1 by blowing-ups, then we can obtain the canonical coordinate systems r_i $(i=0,1,3,4)$.*

Proof. By the following steps, we can resolve the accessible singular locus C_0.

Step 1: We blow up along the curve C_0.

$$X_3^{(2)}=\frac{X_3^{(1)}}{Y_3^{(1)}},\quad Y_3^{(2)}=Y_3^{(1)},\quad Z_3^{(2)}=\frac{Z_3^{(1)}}{Y_3^{(1)}},\quad W_3^{(2)}=W_3^{(1)}.$$

Step 2: We blow up along the surface $\{(X_3^{(2)},Y_3^{(2)},Z_3^{(2)},W_3^{(2)})|X_3^{(2)}=-Z_3^{(2)}W_3^{(2)}+\alpha_0\}$

$$X_3^{(3)}=\frac{X_3^{(2)}+Z_3^{(2)}W_3^{(2)}-\alpha_0}{Y_3^{(2)}},\quad Y_3^{(3)}=Y_3^{(2)},\quad Z_3^{(3)}=Z_3^{(2)},\quad W_3^{(3)}=W_3^{(2)}.$$

By choosing a new coordinate system as

$$(x_0,y_0,z_0,w_0)=(-X_3^{(3)},Y_3^{(3)},Z_3^{(3)},W_3^{(3)}),$$

we can obtain the coordinate system r_0.

For the remaining accessible singular loci, the proof is similar. Thus, we have completed the proof of Proposition 5.2.

PROPOSITION 5.3. *After a series of explicit blowing-ups given in Proposition 5.2, we obtain the smooth projetive* 4*-fold* $\tilde{\mathcal{S}}$ *and a birational morphism* $\varphi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$*. Its canonical divisor* $K_{\tilde{\mathcal{S}}}$ *is given by*

$$K_{\tilde{\mathcal{S}}} = -3\tilde{\mathcal{H}} - \sum_{i=0}^{3} \mathcal{E}_i, \tag{21}$$

where the symbol $\tilde{\mathcal{H}}$ *denotes the proper transform of* $\mathcal{H}$ *by* φ *and* $\mathcal{E}_i$ *denote the exceptional divisors obtained by Step* 1 (see Proof of Proposition 5.2).

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